JOURNAL OF APPROXIMATION THEORY 7, 325–333 (1973)

Determination and Study of Positive Forms on Spaces of Functions, I

GUSTAVE CHOQUET

Department of Mathematics, University of Paris, Paris, France

Communicated by Oved Shisha

Received December 10, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

I. INTRODUCTION

The determination of positive linear forms on a linear space V of real functions defined on a set E, everywhere or almost everywhere with respect to negligible subsets, is a classical topic in analysis. Let us mention, for instance, the moment problem with its extensive literature or the determination of the duals of many interesting ordered Banach spaces V using the fact that every positive form on V is continuous.

In such problems a natural trend is to try and identify positive linear forms with Radon measures on some compact or locally compact space associated with the given set E. This is often possible, especially when dealing with algebras of functions: A good example is the classical moment problem. But it may happen, even in the case of algebras, that a direct use of Radon measures is no longer sufficient; in this case, new tools have to be introduced. Two simple examples will show us the way:

(1) Let V_1 be the algebra of real polynomials $p(x)$ on $E = [0, \infty[$ vanishing at the point $x = 0$; the linear form T_1 on $V_1 : p \to p'(0)$ is positive, and there exists no positive Radon measure μ on E such that $T_1(p) = \mu(p)$ for every $p \in V_1$. But if we define $\bar{p}$ by $\bar{p}(x) = p(x)/x$, we notice that $\bar{p}$ is continuous on E, and that $T_1(p) = \epsilon_0(\bar{p})$, where ϵ_0 is the valuation (or Dirac measure) at 0.

(2) Let V_2 be the linear space of second degree real polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ on $E = [-\infty, \infty]$. The linear form T on $V_2 : p \to a_2$ is positive, and again there is no positive Radon measure μ on E such that

$T(p) = \mu(p)$ for every $p \in V_2$. But if we define $\bar{p}$ by $\bar{p}(x) = p(x)/(x^2 + 1)$, we notice that $\bar{p}$ is continuous on $[-\infty, \infty]$ and that $T_2(p) = \epsilon_\infty(\bar{p})$.

In both cases, by using quotients we have been able to represent T_1 and T_2 in terms of Radon measures ϵ_0 and ϵ_∞. In the first example, the measure was supported by E itself; in the second we had to use a compactification of E.

We will show that one can systematize these ideas, and represent every positive form on V in terms of quotients and Radon measures on some compact space associated with E. More explicitly, we will associate with the couple (E, V) a compact space in which, in some sense, the quotient of any two continuous functions is continuous; such spaces will be called *substonian*, and the linear forms associated with quotients will be called *sub-measures*.

These two notions will prove also very handy in studying extreme positive forms.

Ultrafilters will be used to define substonian spaces associated with couples (E, V); they will be used also to construct many examples of linear spaces V of interest in the study of weakly complete convex cones.

The field of our investigation is rather broad. Indeed, for an ordered linear space V (with $V = V_+ - V_+$) to be identified with a space of real functions on a set E, it is necessary and sufficient that the set of positive linear forms on V separates points of E. Of course, such a generality might be a handicap rather than a reason for interest; however, although our setting is very general, it poses many interesting questions, even for a classical analyst.

For instance, problems studied in the framework of the classical moment problem, such as uniqueness of the Radon measure representing a given linear form, find their equivalent in more general algebras, with the added difficulty of new phenomena: Existence of points where every function in the algebra is zero, or of points where the only values taken are 0 and $\pm\infty$.

Another problem is the following: We have represented positive forms in terms of sub-measures on substonian spaces, which are large compact spaces. Under what general circumstances, is it possible to replace these spaces by smaller ones?

This work began in 1964 at the University of Washington, Seattle, during useful discussions with R. Phelps. I obtained at that time some results concerning extremal positive forms, and I had a vague notion of sub-measures. But only recently, thanks to substonian spaces, could I get a cleaner theory.

II. Substonian Spaces

The linear space of real valued functions on a discrete set E can be identified with the space of continuous mappings of βE (the compact space of ultrafilters on E) into $\overline{\mathbb{R}} = [-\infty, \infty]$, which are finite on the canonical image of E into βE. An analogous procedure can be applied to linear spaces of functions defined on E outside of exceptional sets; but here the compact space βE must be replaced by another one. The essential feature that such a compactification should retain to render possible the definition of sub-measures is the continuity of the quotient of two continuous functions; it is this property that we use now to define substonian spaces.

Let us recall that a *stonian* space is a compact topological space E with the following equivalent properties:

(1) For any pair O_1, O_2 of disjoint open subsets of E, $\bar{O}_1$ and $\bar{O}_2$ are disjoint.

(2) For any open subset O of E, any $f \in \mathscr{C}(O, \overline{\mathbb{R}})$ can be extended into an $\bar{f} \in \mathscr{C}(\bar{O}, \overline{\mathbb{R}})$.

A closed subset of a stonian space is not always stonian. For instance $\beta\mathbb{N}$ is stonian, but not $(\beta\mathbb{N}\backslash\mathbb{N})$. This lack of heredity is often a hindrance in applications; this will not happen with substonian spaces.

Notations 1. For any topological space E, $\mathscr{D}(E)$ will denote the subset of functions in $\mathscr{C}(E, \overline{\mathbb{R}})$ which are infinite only on a nowhere dense (i.e., rare) subset of E; it is not in general a linear space.

For any $f \in \mathscr{D}(E)$, $s(f)$ and $S(f)$ will denote, respectively, the strict support $\{x : f(x) \neq 0\}$ of f, and its closed support $\overline{s(f)}$.

When E is locally compact, for any Radon measure μ on E, $S(\mu)$ will denote also the closed support of μ.

For any $f \in \mathscr{D}(E)$, with $f \neq 0$, $s(f)$ is not empty and the open subset $s'(f)$ of $s(f)$ on which f is finite is everywhere dense in $s(f)$, so that, for any $g \in \mathscr{D}(E)$, the restriction of g/f to $s'(f)$ is defined and belongs to $\mathscr{D}(s'(f))$. As a consequence, if this restriction can be extended to a function $h \in \mathscr{C}(S(f), \bar{R})$, necessarily $h \in \mathscr{D}(S(f))$. This leads us to

DEFINITION 2. We call *substonian* any compact space E such that $\mathscr{D}(E)$ is an algebra and admits sub-quotients in the sense that, for any $f, g \in \mathscr{D}(E)$, g/f belongs to $\mathscr{D}(S(f))$.

In this definition, by "$\mathscr{D}(E)$ is an algebra" we mean that, for any $f, g \in \mathscr{D}(E)$, $(f + g)$ and fg which are defined and finite on an everywhere dense open set, have an extension to E which belongs to $\mathscr{D}(E)$.

We give now handy criteria for a space to be substonian:

THEOREM 3. *If E is a compact space, the following properties are equivalent*:

(1) *E is substonian.*

(2) *Any $f \in \mathscr{C}(E)$ has a sub-inverse (i.e., $1/f \in \mathscr{D}(S(f))$).*

(3) *$f \in \mathscr{C}(E)$, in the neighborhood of any point, is either $\geqslant 0$, or $\leqslant 0$.*

(4) *If ω_1, ω_2 are disjoint open K_σ subsets of E, $\overline{\omega_1} \cap \overline{\omega_2} = \varnothing$.*

(5) *If A_1, A_2 are disjoint K_σ subsets of E, relatively open in $A_1 \cup A_2$, then $\bar{A}_1 \cap \bar{A}_2 = \varnothing$.*

(6) *For any K_σ subset A of E, any $f \in \mathscr{C}(A, \bar{\mathbb{R}})$ can be extended into an $\hat{f} \in \mathscr{C}(\bar{A}, \bar{\mathbb{R}})$.*

(7) *For any open K_σ subset ω of E, any $f \in \mathscr{C}(\omega, \bar{\mathbb{R}})$ can be extended into an $\hat{f} \in \mathscr{C}(\bar{\omega}, \bar{\mathbb{R}})$.*

Proof. (1) $\Rightarrow$ (2) because (2) is weaker than (1).

(2) $\Rightarrow$ (3). Let $f \in \mathscr{C}(E)$ and $a \in E$. If $a \notin S(f)$, f vanishes identically in the neighborhood of a; if $a \in S(f)$ and $f(a) \neq 0$, (3) obviously holds. Finally, if $a \in S(f)$ and $f(a) = 0$, then $(1/f) \in \mathscr{D}(S(f))$ implies that $(1/f)(a) = \pm\infty$, hence f has a fixed sign on $S(f)$ around a; and as f is zero outside $S(f)$, this is true also in E.

(3) $\Rightarrow$ (4). If ω_i $(i = 1, 2)$ is an open K_σ, one can construct by a well-known procedure an $f_i \in \mathscr{C}(E, \mathbb{R})$ which is > 0 on ω_i and zero outside of ω_i. Let $f = f_1 - f_2$; if there exists an $a \in \overline{\omega_1} \cap \overline{\omega_2}$, f does not have a fixed sign around a, in contradiction with (3); hence (4).

(4) $\Rightarrow$ (5). It is sufficient to prove the following result, valid in any compact space E: If A, B are disjoint K_σ subsets of E, which are relatively open in $A \cup B$, there exist disjoint open sets containing, respectively, A and B, which are also K_σ.

Let $A = \bigcup_n A_n$, $B = \bigcup_n B_n$, with A_n and B_n compact, and remember that, in E, any closed set has a base of open neighborhoods which are K_σ. Using the fact that A, B are both open and closed in $A \cup B$, we define recursively open K_σ neighborhoods α_n, β_n in E of A_n, B_n, respectively:

α_1 and β_1 are chosen so that $\overline{\alpha_1} \cap (B \cup \beta_1)$ and $\overline{\beta_1} \cap (A \cup \alpha_1)$ are empty. Suppose now α_p, β_p are defined for $p \leqslant n$, so that $(A \cup \alpha_1 \cdots \cup \alpha_n)$ and $(B \cup \beta_1 \cdots \cup \beta_n)$ are disjoint and closed in their union. Then α_{n+1}, β_{n+1} are taken as any open K_σ neighborhoods of A_{n+1}, B_{n+1}, respectively, such that $\overline{\alpha_{n+1}} \cap (B \cup \beta_1 \cdots \cup \beta_{n+1})$ and $\overline{\beta_{n+1}} \cap (A \cup \alpha_1 \cdots \cup \alpha_{n+1})$ are empty.

The open neighborhoods of A, B we were looking for, are, respectively, $\omega_A = \bigcup_n \alpha_n$ and $\omega_B = \bigcup_n \beta_n$.

(5) $\Rightarrow$ (6). Let $f \in \mathscr{C}(A, \bar{\mathbb{R}})$, where A is a K_σ of E. We want to show that for any $a \in \bar{A}$, f has a single limit value on A at the point a. Suppose that f had two different such limit values λ, μ, and let k be any number strictly between λ and μ. If $A_1 = \{x \in A : f(x) < k\}$ and $A_2 = \{x \in A : f(x) > k\}$, A_1 and A_2

are nonempty and are F_σ of A, and hence are K_σ sets, open in their union. From (5) we conclude that $\overline{A_1}, \overline{A_2}$ are disjoint, in contradiction with the hypothesis that a is in $\overline{A_1}$ and in $\overline{A_2}$.

(6) $\Rightarrow$ (7) because (7) is weaker than (6).

(7) $\Rightarrow$ (1). Let us prove for instance, assuming (7), that $f, g \in \mathscr{D}(E)$ implies $fg \in \mathscr{D}(E)$: The set $\omega = \{x \in E : f(x) \text{ and } g(x) \in \mathbb{R}\}$ is everywhere dense in E and is a K_σ because $\mathbb{R}$ itself is an F_σ of $\overline{\mathbb{R}}$. The product fg is finite and continuous in ω, so that by (7), it has an extension to $\bar{\omega} = E$ which is in $\mathscr{D}(E)$. The proof is analogous for $(f + g)$ and g/f.

Corollary 4. *Every stonian space is substonian.*

Indeed, the basic property of stonian spaces concerns arbitrary disjoint open sets, and hence is stronger than 3.4.

Corollary 5. *Every closed subset of a substonian space is substonian.*

This is an obvious consequence of 3.3, 3.5, or 3.6. For example, $(\beta\mathbb{N}\backslash\mathbb{N})$ is not stonian but it is closed in $\beta\mathbb{N}$, and so substonian.

The following corollary shows that substonian spaces are far from being metrizable.

Corollary 6. (1) *The Alexandrov compactification $\overline{\mathbb{N}}$ of $\mathbb{N}$ is not substonian.*

(2) *A nonisolated point of a substonian space E is never a G_δ of E.*

(3) *Any metrizable closed subset F of a substonian space E is finite.*

Proof. (1) The function $n \to (-1)^n$ on $\mathbb{N}$ does not converge at infinity; hence, by 3.7, $\overline{\mathbb{N}}$ is not substonian.

(2) If $a \in E$ is a nonisolated point of E and a G_δ of E, E contains a subspace homeomorphic to $\overline{\mathbb{N}}$, hence by 6.1 and 5, E is not substonian.

(3) This is a direct consequence of 5 and 6.2 applied to F.

At this point we must remark, as was pointed out by Ajlani, that substonian spaces are exactly, in the class of F-spaces considered by Gillman and Jerison, those which are compact. Indeed, although these authors were led to F-spaces via the study of ideals, one of their characterizations of F-spaces (see 14.25 in [1]) is the following:

"For any $f \in \mathscr{C}(E)$, $\{x : f(x) > 0\}$ and $\{x : f(x) < 0\}$ are completely separated."

For compact E, this condition is equivalent to our 3.3.

These authors have also considered, under the name of *basically disconnec-*

ted spaces (Problems $1H$, $3N$, $4N$ in [1]) another class of topological spaces, which are substonian when they are compact; let us study them briefly.

It is well known that stonian spaces are exactly those compact spaces E for which the ordered space $\mathscr{C}(E, \overline{\mathbb{R}})$ is a complete lattice. Let us say now that a lattice is σ-complete when every denumerable subset of it has an upper and a lower bound. Then:

THEOREM 7. *Let E be a compact space.*

(1) *($\mathscr{C}(E, \overline{\mathbb{R}})$ is a σ-complete lattice) $\Leftrightarrow$ (For any K_σ open $\omega \subset E$, $\bar{\omega}$ is open).*

(2) *These conditions imply that E is substonian; the converse is false.*

Proof. (1) For 7.1, see Problem $3N$ of [1].

(2) The second condition in 7.1 obviously implies 3.4, hence that E is substonian.

To show that the converse is false, it is sufficient to prove that $(\beta\mathbb{N}\backslash\mathbb{N})$, which is substonian, does not satisfy conditions 7.1.

For technical reasons, we will replace $\mathbb{N}$ by $\mathbb{N}^2$. So, let $E = \beta(\mathbb{N}^2)\backslash\mathbb{N}^2$, and let $\Omega = \{$the set of nontrivial ultrafilters supported by one of the sets $n \times \mathbb{N} \subset \mathbb{N}^2\}$. It is a K_σ open subset of E; so it is sufficient to prove that $\bar{\Omega}$ is not open. If it were, as it contains no isolated point, it would be identical with the set of nontrivial ultrafilters on a subset $X \subset \mathbb{N}^2$, where X contains every $n \times \mathbb{N}$ with the possible exception of a finite set. Hence, there would exist a subset Y of X intersecting each $n \times \mathbb{N}$ at exactly one point. As $(\beta Y\backslash Y)$ is open in E and does not intersect Ω, we get a contradiction.

When E satisfies conditions 7.1, it is called *basically disconnected.*

Remark 8. For a compact space, "stonian" is stronger than "basically disconnected", which is stronger than "substonian". Every compact space which is stonian or basically disconnected is totally disconnected; but as Ajlani has remarked, this does not hold for substonian spaces: Indeed, let X be a locally compact space whose point at infinity has a denumerable base (V_n) of connected neighborhoods; if βX is its Stone–Cěch compactification, the compact $E = (\beta X \setminus X)$ is substonian because X is a K_σ (see 14.27 in [1]); moreover, in βX, $E = \bigcap_n \overline{V}_n$, hence E is connected. In particular, this proves that a substonian space cannot always be imbedded in a stonian space.

We will now construct some substonian spaces, which later will prove very useful.

Let $\mathscr{T}$ be a tribe (i.e., stable by denumerable unions and by complementations) of subsets of a given set I. A $\mathscr{T}$-filter is any filter on I with a base in $\mathscr{T}$.

Let $M(\mathscr{T})$ be the linear space of real valued $\mathscr{T}$-measurable functions on I.

Finally, let $\tilde{I}(\mathscr{T})$ be the set of finitely additive measures μ on $\mathscr{T}$, taking only the values 0, 1, and satisfying $\mu(I) = 1$.

For any $\mu \in \tilde{I}(\mathscr{T})$, the set $\{X \in \mathscr{T} : \mu(X) = 1\}$ is a maximal $\mathscr{T}$-filter; and reciprocally, to any maximal $\mathscr{T}$-filter $\mathscr{U}$, the function $\tilde{\mathscr{U}}$ on $\mathscr{T}$ defined by : $\tilde{\mathscr{U}}(X) = 0$ if $X \notin \mathscr{U}$, and $\tilde{\mathscr{U}}(X) = 1$ if $X \in \mathscr{U}$, is an element of $\tilde{I}(\mathscr{T})$; moreover, $\tilde{\tilde{\mathscr{U}}} = \mathscr{U}$ and $\tilde{\tilde{\mu}} = \mu$.

The set $\tilde{I}(\mathscr{T})$ is obviously a closed subset of the compact space $\{0, 1\}^{\mathscr{T}}$, hence $\tilde{I}(\mathscr{T})$ is compact for the topology of simple convergence. If we interpret $\tilde{I}(\mathscr{T})$ as the set of maximal $\mathscr{T}$-filter's by the bijection $\mathscr{U} \to \tilde{\mathscr{U}}$, this compact space has exactly the Stone topology; in this space, the subsets $\tilde{X}(\mathscr{T}) = \{$elements of $\tilde{I}(\mathscr{T})$ supported by $X\}$, where $X \in \mathscr{T}$, make up a base of clopen sets of $\tilde{I}(\mathscr{T})$. We sum up:

LEMMA 9. *The compact space $\tilde{I}(\mathscr{T})$ of finitely additive probability measures on $\mathscr{T}$ with values* 0, 1, *is identifiable with the Stone space of maximal $\mathscr{T}$-filters on I, through the bijection $\mathscr{U} \to \tilde{\mathscr{U}}$.*

It is clear (and well known) that, for any $f \in M(\mathscr{T})$ and $\mathscr{U} \in \tilde{I}(\mathscr{T})$, f has a limit $\tilde{f}(\mathscr{U})$ relative to $\mathscr{U}$, and that $\tilde{f} \in \mathscr{D}(\tilde{I}(\mathscr{T}))$. The mapping $f \to \tilde{f}$ is an algebra homomorphism and as $(f \neq 0) \Rightarrow (\tilde{f} \neq 0)$, this mapping is injective. Let us verify that it is also surjective onto $\mathscr{D}(\tilde{I}(\mathscr{T}))$:

Any $g \in \mathscr{C}(\tilde{I}(\mathscr{T}))$ with a finite range is of the form $\tilde{f}$; using uniform approximation this can be extended to any $g \in \mathscr{C}(\tilde{I}(\mathscr{T}))$ and finally, using truncated functions $g_n = \sup[\inf(g, n), -n]$, this can be extended to any $g \in \mathscr{D}(\tilde{I}(\mathscr{T}))$. We sum up:

PROPOSITION 10. *The mapping $f \to \tilde{f}$ is an (ordered algebra)-isomorphism of $M(\mathscr{T})$ onto $\mathscr{D}(\tilde{I}(\mathscr{T}))$.*

We want to prove now that $\tilde{I}(\mathscr{T})$ is substonian.[1]

THEOREM 11. *For any tribe $\mathscr{T}$ on a set I, the space $\tilde{I}(\mathscr{T})$ of maximal $\mathscr{T}$-filters is basically disconnected, hence substonian.*

Proof. All we have to verify is that, for any uniformly bounded increasing sequence (f_n) in $M(\mathscr{T})$, the sequence $(\tilde{f}_n)$ has an upper bound in $\mathscr{C}(\tilde{I}(\mathscr{T}))$; obviously this upper bound is exactly $\tilde{f}$, where $f = \lim_{n \to \infty} f_n$.

Problem 12. This result raises a question: Is it true, conversely, that every basically disconnected compact space E can be identified with some space $\tilde{I}(\mathscr{T})$?

[1] $\tilde{I}(\mathscr{T})$ is not always stonian: For instance; if I is not denumerable and $\mathscr{T}$ is the tribe generated by denumerable subsets of I.

Let us now try to extend 10 and 11 to tribes coupled with a class of negligible sets.

Let $\mathcal{N}$ be an ideal of $\mathcal{T}$ (i.e., a subset of $\mathcal{T}$, hereditary and stable by finite unions), with $\mathcal{N} \neq \mathcal{T}$; its elements will be called *negligible* sets.

A $(\mathcal{T}, \mathcal{N})$-filter is a $\mathcal{T}$-filter $\mathcal{F}$ such that $(X \in \mathcal{F})$ and $(X = Y \bmod \mathcal{N})$ imply $(Y \in \mathcal{F})$.

Let $M(\mathcal{T}, \mathcal{N})$ be the linear space of classes (mod $\mathcal{N}$) of real valued $\mathcal{T}$-measurable functions, defined $\mathcal{N}$-almost everywhere on I: It can be identified with the quotient of $M(\mathcal{T})$ by its linear subspace of functions which are zero outside of some $X \in \mathcal{N}$.

Finally, let $\tilde{I}(\mathcal{T}, \mathcal{N}) = \{\mu \in \tilde{I}(\mathcal{T}) : \forall X \in \mathcal{N}, \mu(x) = 0\}$; it is closed in $\tilde{I}(\mathcal{T})$ and is identical, through the mapping $\mathcal{U} \to \tilde{\mathcal{U}}$, to the set of maximal $(\mathcal{T}, \mathcal{N})$-filters. We sum up:

LEMMA 13. *The space $\tilde{I}(\mathcal{T}, \mathcal{N})$ of maximal $(\mathcal{T}, \mathcal{N})$-filters is a closed subset of $\tilde{I}(\mathcal{T})$, and hence substonian.*

For any $f \in M(\mathcal{T})$, let now $\tilde{f}_{\mathcal{N}}$ be the restriction of $\tilde{f}$ to $\tilde{I}(\mathcal{T}, \mathcal{N})$; obviously $(f = g \bmod \mathcal{N}) \Rightarrow (\tilde{f}_{\mathcal{N}} = \tilde{g}_{\mathcal{N}})$ so that we get in fact a mapping of $M(\mathcal{T}, \mathcal{N})$ into $\mathcal{D}(\tilde{I}(\mathcal{T}, \mathcal{N}), \overline{\mathbb{R}})$; but this mapping is not interesting in general because it can fail to be injective, and also because $\tilde{f}_{\mathcal{N}}$ can be identically ∞. This is the case, for instance, when $I = \mathbb{N}$, $\mathcal{T} = \mathcal{P}(\mathbb{N})$, and $\mathcal{N}$ is the set of finite subsets of $\mathbb{N}$; for any sequence $n \to f(n)$ such that $\lim_{n\to\infty} f(n) = \infty$, $\tilde{f}_{\mathcal{N}}$ is identically ∞ on $\tilde{I}(\mathcal{T}, \mathcal{N}) = (\beta\mathbb{N} \setminus \mathbb{N})$.

Fortunately, this cannot happen when $\mathcal{N}$ is a σ-ideal (i.e., stable by denumerable unions):

THEOREM 14. *When $\mathcal{N}$ is a σ-ideal of the tribe $\mathcal{T}$, with $\mathcal{N} \neq \mathcal{T}$, the mapping $f \to \tilde{f}_{\mathcal{N}}$ is an (ordered algebra)-isomorphism of $M(\mathcal{T}, \mathcal{N})$ onto $\mathcal{D}(\tilde{I}(\mathcal{T}, \mathcal{N}))$.*

Proof. Let $f \in M(\mathcal{T}, \mathcal{N})$, let $\phi \in M(\mathcal{T})$ be an element of the class f, and let $\mathcal{U} \in \tilde{I}(\mathcal{T}, \mathcal{N})$. For any clopen neighborhood $\tilde{X}(\mathcal{T})$ of $\mathcal{U}$ in $\tilde{I}(\mathcal{T})$, X is not negligible, so there exists $y \in \mathcal{T}$, with $Y \notin \mathcal{N}$ and $Y \subset X$, such that ϕ is bounded on Y; hence $\tilde{f}_{\mathcal{N}}$ is bounded on $\tilde{y}(\mathcal{T}) \cap \tilde{I}(\mathcal{T}, \mathcal{N})$. This proves that $\tilde{f}_{\mathcal{N}} \in \mathcal{D}(\tilde{I}(\mathcal{T}, \mathcal{N}))$.

It follows immediately that the mapping $f \to \tilde{f}_{\mathcal{N}}$ is an (ordered algebra)-homomorphism. It is injective because if $f \neq 0$, a similar reasoning proves that on some $y \in \mathcal{T}$, with $y \notin \mathcal{N}$, ϕ has its range in a closed interval $[u, v]$ not containing zero; hence $\tilde{f}_{\mathcal{N}}$ on $\tilde{y}(\mathcal{T})$ has also its range in $[u, v]$, and so $\tilde{f}_{\mathcal{N}} \neq 0$.

To prove its surjectivity is equivalent, by Proposition 10, to proving that every $g \in \mathcal{D}^+(\tilde{I}(\mathcal{T}, \mathcal{N}))$ is the restriction of some element of $\mathcal{D}^+(\tilde{I}(\mathcal{T}))$; this is obvious when g is bounded. Suppose now that g is arbitrary, and let $g_n = \inf(g, n)$; there exists a unique $f_n \in M(\mathscr{T}, \mathscr{N})$ with $(\tilde{f}_n)_{\mathscr{N}} = g_n$.

As $f_p = \inf(f_q, p)$ whenever $p \leqslant q$, one can define by recursion a sequence (ϕ_n), $\phi_n \in M(\mathscr{T})$, where ϕ_n is an element of the class f_n, such that $\phi_p = \inf(\phi_q, p)$ when $p \leqslant q$.

The function $\phi = \lim_{n\to\infty} \phi_n$ is such that $\phi_n = \inf(\phi, n)$ for each n, and so the restriction of $\tilde{\phi}$ to $\tilde{I}(\mathscr{T}, \mathscr{N})$ is exactly f.

Remark 15. (1) The aim of what follows is the study of positive linear forms on subspaces V of a given $M(\mathscr{T}, \mathscr{N})$. When $\mathscr{N}$ is a σ-ideal of $\mathscr{T}$, Theorem 14 shows that our study is equivalent to the study of positive forms on the image $\tilde{V}_{\mathscr{N}}$ of V in $\mathscr{D}(\tilde{I}(\mathscr{T}, \mathscr{N}))$.

But when $\mathscr{N}$ is not a σ-ideal, we have to come back to $\tilde{V}$ and study positive forms on this subspace of $\mathscr{D}(\tilde{I}(\mathscr{T}))$ which are zero for every negligible element $\tilde{f}$ of $\tilde{V}$.

Instead of dealing only with spaces $\tilde{I}(\mathscr{T}, \mathscr{N})$ or $\tilde{I}(\mathscr{T})$, we will consider more generally arbitrary substonian spaces: This will be both simpler and more general.

(2) Although what follows concerns linear subspaces $V \subset \mathscr{D}(E)$ whose E is substonian, most of our examples will concern subspaces of some $M(\mathscr{T}, \mathscr{N})$; we shall not repeat everywhere that Theorem 14 establishes a translation of these examples in terms of substonian spaces.

Reference

1. L. Gillman and M. Jerison, "Rings of Continuous Functions," Van Nostrand, Princeton, NJ, 1960.